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The Principle of Reciprocity

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Abstract

It is shown that the validity of the principle of reciprocity for arbitrary motion implies that the rate of a moving clock departs from the (instantaneous) special relativistic value by a term $-x\ddot{x}/2c^2$ where x is the distance of the clock. Through this effect the cosmological red shift can be understood as arising from the outward acceleration of radiating atoms in galactic gravitational fields. In this way the relation $(GM/2cL^2) \sim (1/H)$ between Newton's constant of gravitation G, the mass M and linear dimension L of a typical galaxy and Hubble's constant H is derived. This relation is verified by present-day observations. Implications of these considerations for quasars are briefly discussed.

As a possible test of this theory it is suggested that the spectra of galaxies be searched for the presence of blue-shifted lines which are expected to be fainter by three to four magnitudes in comparison with the red-shifted lines.

1. *Introduction*

Some time ago we discussed (Khan, 1968) the problem of non-uniform relative motion between two elementary observers (monads). We envisaged the possibility that under certain circumstances a reciprocity principle may be valid for the space-time observations of each carried out on the other. More precisely, we postulated the existence, under special conditions, of frames of reference L and L' fixed relative to the observers A and A', respectively, such that the space-time observations on A carried out in the frame of reference L' coincide with the observations on A' carried out in L. Underlying this scheme of ideas are the notions of ideal clocks and rods. Indeed one may violate the principle by a bad choice of clocks and rods. This viewpoint is completely opposed to the viewpoint of Einstein's general relativity according to which all co-ordinate frames are equivalent and there are no ideal clocks and rods. As a justification of our viewpoint we may refer to the actual situation in experimental physics where one is certainly aware of actual and possible clocks that are truly fundamental in the sense of simplicity of essential constitution. We have in mind the present-day cesium clocks and the clocks of not too distant future based on the activation of a submultiple generator by means of the Mössbauer radiation (Brillouin, 1970). Indeed the whole of experimental physics is motivated by the belief that there are certain ideal measuring devices which can in practice be realised to an arbitrarily close approximation by the gradual perfection of experimental technique. Furthermore, a close analysis of the notion of an ideal clock from the point of view of an experimental physicist reveals that it leans heavily on the notion of an ideal rod. This is seen from the fact that recoil effects in an ideal clock associated with radiation emission or other modes of interaction have to be rectified by imbedding the clock in an ideal framework. In the case of Mössbauer clocks, the ideal framework is realised by the crystalline lattice. It should be noted that we have carefully avoided the phrases 'rigid framework' and 'rigid rod' which in ordinary language have different connotations.

We have specified (Khan, 1968) the conditions under which the reciprocity principle may be expected to hold for the world pictures of the two observers. The conditions are just the absence of electromagnetic fields extraneous to the system of two elementary observers (particles). Any electromagnetic fields intrinsic to the system of two particles, however, are assumed not to affect the applicability of the reciprocity principle; nor does the presence of gravitational fields (extraneous or intrinsic) limit the application of the reciprocity principle in the form postulated above. In our thought experiment of two particles (unperturbed by extraneous electromagnetic fields) the relative motion is envisaged as arising from evolution in time of the electromagnetic field intrinsic to the system. The changing electromagnetic field not only produces the observed relative motion but is also supposed to interact with the ideal clocks and rods (considered as electromagnetic devices) carried by the two observers (particles) in such a way that the principle of reciprocity holds for the two world pictures. Thus, by invoking the principle of reciprocity for the system, we may forget about fields intrinsic to the system. They are there just to ensure the validity of the reciprocity principle for arbitrary motion in a self-consistent way. We may here draw an analogy with the situation in pre-special-relativistic physics highlighted by the work of Lorentz and Fitzgerald demonstrating the contraction of rods in the presence of 'ether drift'. With the advent of special relativity (which is equivalent to reciprocity for uniform relative motion) it was realised that the ether is a redundant concept and its role is merely to ensure the validity of special relativity for arbitrary uniform motions. Similarly, one may hope that if the programme of reciprocity could be carried through to a system of arbitrary number of particles, the concept of the electromagnetic field itself would become redundant. However, at present great mathematical difficulties are being encountered in the fulfillment of such a programme and we shall not attempt to discuss the general many-particle system here.

An important consequence of the reciprocity principle obtained in our paper is that the rate of a clock in accelerated motion deviates from the instantaneous (special) relativistic value by terms of order $x\ddot{x}/c^2$ where x is the distance of the clock from the point of observation. Now, for ordinary the ground of our use of the notion of an ideal rod for which he has no sympathy, we shall present an exactly soluble motion in which we avoid using an *a priori* notion of length and work with time measurements only. This latter example, presented in Section 3, furthermore has the attractive continuity properties which are lacking in the first example. In Section 4 we tackle the general two-particle system using a powerful technique suggested by the work of Jabotinsky and Lüntz in the context of Abel's functional equation (Hadamard, 1944). The solution thus obtained is in the form of a power series. This method enables us to bypass the ambiguities of the calculus for general motions proposed in our earlier paper.

2. *A Thought Experiment*

Suppose a system of two particles (observers) A, A' is subject only to internal electromagnetic forces. According to the schemata described in the Introduction, the evolution in time of the electromagnetic field intrinsic to the system is reflected in the development of relative motion between the two particles and the behaviour of ideal clocks and rods carried by the observers consistent with the principle of reciprocity. Suppose now that A observes (in its frame of reference L) A' to move (radially) with uniform velocity v_1 up to a distance d where the velocity is suddenly changed to a new value v_2 ($\neq v_1$)—this value being maintained in the subsequent motion. One may readily imagine such a change brought about by a change in the electromagnetic field of the system—e.g., emission of a photon by A' in the frame of reference L. From the principle of reciprocity, A' would also observe A to undergo the same motion in its frame of reference L'. If for the moment we believe naïvely that at the instant of velocity transition of A' (in the frame L) the clock carried by A' records an instant and does not make a jump we are immediately led to the conclusion that A' observes A in its frame to move with uniform velocity v_1 up to a distance $d\sqrt{[1-(v_1^2/c^2)]}$ where A performs a sudden jump to a distance $d\sqrt{[1-(v_2^2/c^2)]}$ before resuming motion with uniform velocity v_2. This motion is clearly very different from that of A' in L. Thus our naïve assumption about the clock carried by A' not undergoing a sudden jump at the instant of velocity transition is in contradiction with the principle of reciprocity. Having demonstrated the necessity for the 'time jump' we shall calculate its magnitude. This can easily be done by noting that it must be such as just to allow observation in L' of the motion of A from a distance

$$r = d\sqrt{[1-(v_1^2/c^2)]}$$

to $r = d$ with uniform velocity v_1 followed by the motion from $r = d$ to $r = d\sqrt{[1-(v_2^2/c^2)]}$ with uniform velocity v_2. Thus it is compounded of a time interval

$$\frac{d}{v_1}\left[1-\sqrt{\left(1-\frac{v_1^2}{c^2}\right)}\right] \gtrless 0 \qquad \text{for } v_1 \gtrless 0 \tag{2.1}$$

the ground of our use of the notion of an ideal rod for which he has no sympathy, we shall present an exactly soluble motion in which we avoid using an *a priori* notion of length and work with time measurements only. This latter example, presented in Section 3, furthermore has the attractive continuity properties which are lacking in the first example. In Section 4 we tackle the general two-particle system using a powerful technique suggested by the work of Jabotinsky and Lüntz in the context of Abel's functional equation (Hadamard, 1944). The solution thus obtained is in the form of a power series. This method enables us to bypass the ambiguities of the calculus for general motions proposed in our earlier paper.

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$$\frac{d}{v_1}\left[1-\sqrt{\left(1-\frac{v_1^2}{c^2}\right)}\right] \gtrless 0 \qquad \text{for } v_1 \gtrless 0 \tag{2.1}$$

followed by a time interval

$$-\frac{d}{v_2}\left[1-\sqrt{\left(1-\frac{v_2^2}{c^2}\right)}\right] \lessgtr 0 \qquad \text{for } v_2 \gtrless 0 \tag{2.2}$$

The total time interval is thus given by $\Delta\{d/v(\sqrt{[1-(v^2/c^2)]}-1)\}$ where Δ defines the change in the argument at the instant of velocity transition.

We should like to remark here that the appearance of negative time intervals in the foregoing argument does not force us to abandon the fundamental requirement of observers arranging events presented to their consciousness in an increasing time sequence. An observer endowed with a human-like consciousness would simply ignore the negative time intervals. For example, the motion from a distance $d\sqrt{[1-(v_1^2/c^2)]}$ to $r=d$ with uniform velocity v_1 (>0) followed by the motion from $r=d$ to $r=d\sqrt{[1-(v_2^2/c^2)]}$ with uniform velocity v_2 (>0) encountered above would just appear to such an observer as the motion from a distance $r=d\sqrt{[1-(v_1^2/c^2)]}$ to $r=d$ with velocity v_1 undergoing a change of velocity from v_1 to v_2 at $r=d$. It is now clear that the two sets of observations are completely identical in conformity with the principle of reciprocity and, in particular, the time jump on the clock carried by A at the instant of velocity transition in the frame L' is also given by $\Delta\{d/v(\sqrt{[1-(v^2/c^2)]}-1)\}$. Thus to calculate the rate difference (as seen by A or A') of the two clocks in uniform relative acceleration in the limit of zero relative velocity we have only to differentiate the expression in { } and take the limit $v \to 0$, i.e. the rate difference†

$$=\frac{d}{dt}\left\{\frac{d}{v}\sqrt{\left(1-\frac{v^2}{c^2}\right)}-\frac{d}{v}\right\}\Bigg|_{v\to 0}$$

$$=\left[\sqrt{\left(1-\frac{v^2}{c^2}\right)}-1-\frac{d\dot{v}}{v^2}\frac{1-\sqrt{(1-v^2/c^2)}}{\sqrt{(1-v^2/c^2)}}\right]_{v\to 0}=-\frac{d\dot{v}}{2c^2} \tag{2.3}$$

It is precisely this expression for the rate difference which gives the relation $(1/H)\sim(GM/2cL^2)$ between Newton's constant of gravitation, Hubble's constant, the velocity of light and the characteristic ratio M/L^2 for galaxies.‡

† This calculation of the rate difference is expected to be valid only in the limit $v \to 0$ for the case of uniform acceleration because we have derived the expression for the time jump $\Delta\{(d/v)\sqrt{(1-v^2/c^2)}-(d/v)\}$ under the assumption of instantaneous acceleration (sudden change of velocity). In Section 3, we have studied the case of uniform (quasi) acceleration and obtained independent confirmation of equation 2.3 (see equation 3.15).

‡ The relation $(1/H)\sim(GM/2cL^2)$ is derived as follows. The light received from galaxies is predominantly from the parts facing us, which are in a state of acceleration GM/L^2 *away* from us. Using equation (2.3),

$$\text{the } red\text{shift} = d.\left(\frac{GM}{L^2}\right)\frac{1}{2c^2}$$

where d is the distance of a galaxy. Thus the Hubble constant H appearing in the empirical redshift distance relation (redshift $=(d/cH)$) is given by

$$\frac{1}{H}\sim\frac{GM}{2cL^2}$$

Notice that we have assumed the galaxies at rest relative to one another. No *ad hoc* hypothesis of expanding universe is needed to understand the cosmological reshift.

Suppose now that in the above thought experiment $v_1 > 0$, $v_2 < 0$ so that the two observers meet again. Let us see what happens to the clock carried by A' from the point of view of observer A. If the two clocks agree at the start of the outward journey then just before the instant of velocity transition the clock carried by A' would be lagging behind by $d/v_1\{1 - \sqrt{[1-(v_1^2/c^2)]}\}$. Just after the instant of velocity transition the clock would be leading by

$$\Delta\left\{\frac{d}{v}\left[\sqrt{\left(1-\frac{v^2}{c^2}\right)}-1\right]\right\}-\frac{d}{v_1}\left[1-\sqrt{\left(1-\frac{v_1^2}{c^2}\right)}\right]$$
$$=\frac{d}{v_2}\left[\sqrt{\left(1-\frac{v_2^2}{c^2}\right)}-1\right]=\frac{d}{|v_2|}\left[1-\sqrt{\left(1-\frac{v_2^2}{c^2}\right)}\right]. \quad (2.3)$$

This is precisely equal to the time lag in the clock during the inward journey with uniform velocity $|v_2|$. Thus the two clocks would agree on meeting again and the so-called twin paradox is non-existent. The non-existence of the twin paradox is not dependent on the peculiarities of the relative motion but is an essential feature of all relative motions consistent with the principle of reciprocity, as will become abundantly clear from consideration of the example discussed below.

3. *An Example of Exactly Soluble Motion*

The discontinuous features present in the thought experiment described above, although not very serious, are avoided in the following example. Furthermore, it is possible to avoid speaking of length measurements and work entirely in terms of time measurements with light signals as Synge† would require.

Suppose that in the thought experiment described above A' moves in the frame L according to the equation of motion

$$\frac{d}{dt}\frac{v}{\sqrt{(1-v^2)}}=a$$

where v is the instantaneous velocity and a is a constant. Integration gives

$$x=\frac{1}{a}\sqrt{[1+(at+b)^2]}+c \quad (3.1)$$

where x is the distance of A' from the origin in L which can in principle be measured entirely in terms of time measurements with light signals. In equation (3.1), b, c are constants of integration.

We may represent the observations of A in the frame L with the help of world lines drawn in the space-time of A, as shown in Fig. 1. These observations are assumed to be carried out entirely by means of light signals and time measurements performed at A (see Fig. 1). Thus the time of the event P' is $t=\frac{1}{2}(t_1+t_2)$ and its (radial) distance is $x=\frac{1}{2}(t_2-t_1)$ taking the velocity

† See footnote, p. 385.

of light equal to unity. We have confined ourselves to the case of relative motion in one spatial dimension as before. We should like to point out here that there is nothing absolute about the space-time diagram presented here in the sense that it is not necessarily the picture of observations carried out by A' in its own space time. The picture in the space-time of A' is obtained by interchanging the labels A, A' in the figure.

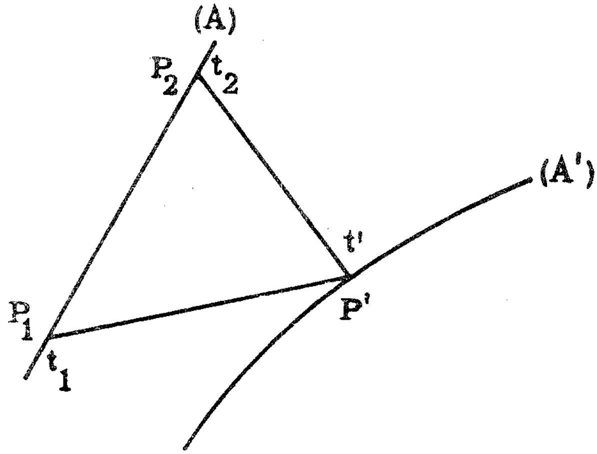

Figure 1.—Space-time picture in the frame L.

Thus we have

$$t_2 = t + x = t + \frac{1}{a}\sqrt{[1 + (at + b)^2]} + c \tag{3.2}$$

$$t_1 = t - x = t - \frac{1}{a}\sqrt{[1 + (at + b)^2]} - c \tag{3.3}$$

Equations (3.2) and (3.3) determine t_2 as a function of t_1. Suppose t' is the time recorded on the clock carried by A' at the instant when the light signal originating at P arrives (at time t in the frame L). Our problem is to determine t' as a function of t or, more elegantly, as a function of t_1. Let $t' = \phi(t_1)$. From the principle of reciprocity $t_2 = \phi(t')$ so that $t_2 = \phi(\phi(t_1))$. Thus the problem reduces to that of finding an iterate of order $\frac{1}{2}$ for the function $t_2(t_1)$ defined by equations (3.2) and (3.3).

Define α so that

$$(c - \alpha)^2 = \frac{1 + b^2}{a^2}$$

Then

$$\frac{1}{t_2 + \alpha} = \frac{t + c - \alpha - (1/a)\sqrt{[1 + (at + b)^2]}}{2[c - \alpha - (b/a)]\,t} \tag{3.4a}$$

$$\frac{1}{t_1+\alpha}=\frac{t-(c-\alpha)+(1/a)\sqrt{[1+(at+b)^2]}}{-2[c-\alpha+(b/a)]\,t} \tag{3.4b}$$

so that

$$\left(c-\alpha-\frac{b}{a}\right)\frac{1}{t_2-\alpha}+\left(-c+\alpha-\frac{b}{a}\right)\frac{1}{t_1+\alpha}=1$$

i.e.,

$$t_2=\frac{(c-b)\,t_1+\alpha(2c-\alpha)}{t_1+c+(b/a)}=\frac{[c-(b/a)]\,t_1+c^2-[(1+b^2)/a^2]}{t_1+c+(b/a)} \tag{3.5}$$

This is the so-called Möbius transformation. A Möbius transformation has the interesting property of preserving its character under iteration. Thus to solve

$$\phi(\phi(t_1))=\frac{[c-(b/a)]\,t_1+c^2-[(1+b^2)/a^2]}{t_1+c+(b/a)} \tag{3.6}$$

we try

$$\phi(t_1)=\frac{pt_1+q}{t_1+r} \tag{3.7}$$

then

$$\phi(\phi(t_1))=\frac{p[(pt_1+q)/(t_1+r)]+q}{[(pt_1+q)/(t_1+r)]+r}=\frac{(p^2+q)\,t_1+q(p+r)}{(p+r)\,t_1+q+r^2} \tag{3.8}$$

therefore

$$\frac{p^2+q}{p+r}=c-\frac{b}{a} \tag{3.9a}$$

$$q=c^2-\frac{1+b^2}{a^2} \tag{3.9b}$$

$$\frac{q+r^2}{p+r}=c+\frac{b}{a} \tag{3.9c}$$

which can be solved for p, q, r.

To show the non-existence of the twin paradox we let $c=-\sqrt{(1+b^2)}/a$ so that $x=0$ when $t=0$. Thus equation (3.9b) gives $q=0$. Now $x=0$ also when $t=-2b/a$. For this value of t we have from equations (3.1) and (3.2) $t_1=t_2=-2b/a$ and

$$\phi\left(-\frac{2b}{a}\right)=\frac{p(-2b/a)}{-(2b/a)+r}$$

Since q vanishes equations (3.9a) and (3.9c) give

$$\frac{p^2}{p+r}=c-\frac{b}{a},\qquad \frac{r^2}{p+r}=c+\frac{b}{a} \tag{3.10}$$

so that $p - r = -2b/a$, and we have

$$\phi\left(-\frac{2b}{a}\right) = -\frac{2b}{a} \tag{3.11}$$

i.e., the two clocks would agree when they meet again.

Let us now calculate the rate (R) of the clock attached to A' when it is stationary in the frame L. Clearly A' has zero velocity when $t = -b/a$ and its distance from A is $1/a + c$.

$$R = \left.\frac{d\phi(t_1)}{dt_1}\right|_{t=-b/a} = \left.\frac{pr - q}{(t_1 + r)^2}\right|_{t_1=-b/a-1/a-c} \tag{3.12}$$

where we have used the equation $t_1 = t - x$. Hence,

$$R = \frac{pr - q}{(-(b/a) - (1/a) - c + r)^2} \tag{3.13}$$

From equations (3.9a), (3.9b) and (3.9c) we get two sets of values for p, q, r, viz.

$$\begin{aligned} p &= \pm\frac{1}{a} + c - \frac{b}{a} \\ q &= c^2 - \frac{1 + b^2}{a^2} \\ r &= \pm\frac{1}{a} + c + \frac{b}{a} \end{aligned} \tag{3.14}$$

Thus there seems to be an apparent non-uniqueness of the solution. However, this problem is easily sorted out by the physical requirement that R be finite. This picks out the solution with the lower sign and we get

$$R = \tfrac{1}{2}(1 - ca) = 1 - \tfrac{1}{2}d.a \tag{3.15}$$

where $d = 1/a + c$ is the distance of the clock. The term $-\frac{1}{2}d.a$ is precisely the term $-d\dot{v}/2c^2$ obtained in the thought experiment of Section 2 in the limit $v \to 0$. Note that we have taken the velocity of light $c = 1$ and in our example $\dot{v} = a$.

4. *Arbitrary Relative Motion in One Dimension*

We shall now discuss arbitrary relative motion in one dimension. In the terminology of the diagram introduced in Section 3 let

$$x = a_0 + a_1 t + a_2 t^2 + \ldots$$

so that

$$t_2 = a_0 + (1 + a_1)t + a_2 t^2 + a_3 t^3 + \ldots \tag{4.1}$$

$$t_1 = -a_0 + (1 - a_1)t - a_2 t^2 - a_3 t^3 - \ldots \tag{4.2}$$

We should like to remark here that the condition $(dx/dt) < 1$ need not bother us at this stage. It can in principle be built into the theory through the formulation of a dynamical law of motion which turns out to be of teleological character in the sense that possible solutions of equations of motion occur implicitly in the formulation of the equations (see Section 5B). At present we are interested in the purely kinematical aspects of the two-particle system.

Equations (4.1) and (4.2) determine t_2 as a function of t_1. Let $t_2 = \chi(t_1)$. To determine the behaviour of the clock carried by A' let $t' = \phi(t_1)$ so that from the principle of reciprocity $t_2 = \phi(t')$ and

$$\phi(\phi(z)) = \chi(z) \tag{4.3}$$

Our problem then is the determination of ϕ satisfying equation (4.3). We shall first reduce this problem to one capable of solution as power series in z. This is done with the help of the following lemma:

Given

$$\phi(\phi(z)) = \chi(z)$$

define

$$\tilde{\chi}(z) = \chi(z+a) - a, \qquad \tilde{\phi}(z) = \phi(z+a) - a$$

then

$$\tilde{\phi}(\tilde{\phi}(z)) = \tilde{\chi}(z)$$

The proof of this is very straightforward. A general form of this lemma is stated in Appendix A.

Now if we take $a = a_0$ in the above lemma, $\tilde{\chi}(z)$ is given by the parametric equations

$$\tilde{\chi}(z) = (1 + a_1)\, t + a_2 t^2 + a_3 t^3 + \ldots \tag{4.4}$$

$$z = (1 - a_1)\, t - a_2 t^2 - a_3 t^3 - \ldots \tag{4.5}$$

The various steps involved in the solution of the problem thus become: (i) Finding the inverse function (iterate of order -1) of the function of t on the right-hand side of equation (4.5). The problem of iterates of arbitrary order is discussed below. (ii) Substituting the inverse function (obtained as a power series in z) in the right-hand side of equation (4.4) to obtain $\tilde{\chi}(z)$ as a power series in z (with no constant term). (iii) Determining the iterate of order $\frac{1}{2}$ of the function $\tilde{\chi}(z)$ obtained above as a power series in z. This gives us the function $\tilde{\phi}(z)$ satisfying $\tilde{\phi}(\tilde{\phi}(z)) = \tilde{\chi}(z)$. (iv) The function $\phi(t)$ (describing the behaviour of the clock A' in the frame L) is evaluated using $\phi(t) = \tilde{\phi}(t - a_0) + a_0$. Thus in the above scheme a central role is played by the solution (in power series) of the general problem of iterates of arbitrary order for a function given as a power series (with no constant term). This solution will now be presented. Let us denote the iterate of order n of a function $\phi(z)$ by $\phi_n(z)$ so that $\phi_{n+1}(z) = \phi(\phi_n(z))$. By induction we have

$$\phi_n(\phi_{m-n}(z)) = \phi_m(z) \tag{4.6}$$

We shall now let m, n become *continuous* variables. It can then be shown that equation (4.6) implies the following partial differential equation (see Appendix B):

$$\frac{\partial^2 \phi_n(x)}{\partial n^2}\frac{\partial \phi_n(x)}{\partial x} - \frac{\partial^2 \phi_n(x)}{\partial n\,\partial x}\frac{\partial \phi_n(x)}{\partial n} = 0 \tag{4.7}$$

The above equation gives the following integral:

$$\frac{[\partial \phi_n(x)]/\partial n}{[\partial \phi_n(x)]/\partial x} = \rho(x) \tag{4.8}$$

where $\rho(x)$ is an arbitrary function of x.

Let

$$\phi_n(x) = c_1(n)\,x + c_2(n)\,x^2 + \ldots \tag{4.9}$$

$$\rho(x) = \rho_1 . x + \rho_2\,x^2 + \rho_3\,x^3 + \ldots \tag{4.10}$$

Then equation (4.8) gives

$$\left.\begin{aligned} c_1'(n) &= \rho_1\,c_1(n)\\ c_2'(n) &= 2\rho_1\,c_2(n) + \rho_2\,c_1(n)\\ c_3'(n) &= 3\rho_1\,c_3(n) + 2\rho_2\,c_2(n) + \rho_3\,c_1(n)\\ c_4'(n) &= 4\rho_1\,c_4(n) + 3\rho_2\,c_3(n) + 2\rho_3\,c_2(n) + \rho_4\,c_1(n)\end{aligned}\right\} \tag{4.11}$$

$\ldots$ etc.,

where the prime denotes differentiation with regard to n. Thus the problem reduces to that of solving an infinite set of ordinary differential equations. The boundary condition $\phi_0(x) = x$ gives $c_1(0) = 1$, $c_i(0) = 0$ for $i \neq 1$. Using these we obtain

$$c_1(n) = e^{\rho_1 n}$$

$$c_2(n) = \frac{\rho_2}{\rho_1}(e^{2\rho_1 n} - e^{\rho_1 n})$$

$$c_3(n) = \frac{1}{2\rho_1}\left(\rho_3 + \frac{2\rho_2^2}{\rho_1}\right)e^{3\rho_1 n} - \frac{2\rho_2^2}{\rho_1^2}e^{2\rho_1 n} - \frac{1}{2\rho_1}\left(\rho_3 - \frac{2\rho_2^2}{\rho_1}\right)e^{\rho_1 n}$$

$$c_4(n) = \frac{1}{3\rho_1}\left(\rho_4 + \frac{4\rho_2\,\rho_3}{\rho_1} + \frac{3\rho_2^3}{\rho_1^2}\right)e^{4\rho_1 n} - \frac{3\rho_2}{2\rho_1^2}\left(\rho_3 + \frac{2\rho_2^2}{\rho_1}\right)e^{3\rho_1 n}$$

$$+ \frac{\rho_2}{\rho_1^3}(3\rho_2^2 - \rho_1\,\rho_3)\,e^{2\rho_1 n} - \frac{1}{3\rho_1}\left(\rho_4 - \frac{7\rho_2\,\rho_3}{2\rho_1} + \frac{3\rho_2^3}{\rho_1^2}\right)e^{\rho_1 n}$$

$\ldots$ etc. (4.12)

The constants $\rho_1, \rho_2, \dots$ can now be determined from the second boundary condition: $\phi_1(x) = \phi(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots$ so that $c_k(1) = a_k$. Thus

$$\rho_1 = \log_e(a_1)$$

$$\rho_2 = \frac{a_2}{a_1{}^2 - a_1} \log_e(a_1)$$

$$\rho_3 = \frac{2(a_3 a_1 - a_2{}^2)}{a_1{}^2(a_1{}^2 - 1)} \log_e(a_1)$$

$$\rho_4 = \frac{3a_1{}^2(a_1+1)a_4 - a_1 a_2 a_3(8a_1+7) + a_2{}^3(5a_1+4)}{a_1{}^3(a_1{}^2-1)(a_1{}^2+a_1+1)} \log_e(a_1)$$

$$\dots \text{etc.} \tag{4.13}$$

The solution of our problem concerning the iterates of arbitrary order (n) of the function $\phi(x)$ (expressible as a power series in x with no constant term) is now complete. In particular the inverse function is obtained by setting $n = -1$ in the above solution, and the solution (ψ) of $\psi(\psi(x)) = \phi(x)$ is obtained by setting $n = \frac{1}{2}$.

5. *Discussion and Remarks*

(A) Let us recapitulate the main results established so far. We have demonstrated that the rate of a clock depends on its acceleration and distance. The rate has been shown to depart from the instantaneous special relativistic value by terms of order $x\ddot{x}/c^2$ where x is the distance of the clock from the point of observation. Such a term in the expression for the rate of a clock has observable consequences in cosmological situations, viz., the so-called cosmological red-shift is claimed to be due to this effect. We have obtained in this manner the following relationship:

$$\frac{GM}{2cL^2} \sim 1/H \tag{5.1}$$

where G is the newtonian constant of gravitation, M is the mass and L the linear dimension of a typical galaxy. Substituting the known values of M and L for the galaxy (Allen, 1963) into the expression on the left-hand side of equation (5.1), we get for the Hubble constant (H) the value 10^{16}–10^{18} seconds. The uncertainty in the figure is due to the uncertainty in the effective linear dimension L of the galaxy. The value accepted at present of the Hubble constant is $(4{\cdot}1 \pm 2)\,10^{17}$ seconds (Sandage, 1958, 1968).

As a test of this theory we suggest a search for possible blue-shifted lines in the spectra of galaxies. These blue-shifted lines are expected to be present in the light originating from those parts of the galaxies which are facing away from us. Due to the absorption caused by the intervening mass of galactic dust, the blue-shifted lines are expected to be fainter by three to four magnitudes in comparison with the red-shifted lines.†

We have suggested that the anomalously large red shifts of the quasi-stellar objects is not due to their distances being very large, rather it is a consequence of the characteristic ratio M/L^2 for these objects being large in comparison with that for the ordinary galaxies. Thus the puzzle of extreme brightness of these objects is resolved. Recently Shapiro (Shapiro, 1971) has presented some observations of relative motion between the two components of the quasi-stellar source 3C 279 which clearly show that the source cannot be at cosmological distances, for this would imply that the two components are flying apart at a velocity greater than the velocity of light!

(B) Finally, we should like to add some remarks concerning the formulation of possible dynamics in the present approach. We wish to emphasise here that the following remarks are of a purely tentative character. The possibility we suggest is the following. The appropriate forces in a two-particle system are to be determined self-consistently through a generalisation of the newtonian equations of motion, viz., $(d/dt)(mv/\phi(t)) = F(x)$ where $\phi(t)$ is the function describing the behaviour of the clock moving according to the law of motion following from the above differential equation. The clock function $\phi(t)$ can in principle be determined from the law of motion using the general method of Section 4. Thus we obtain a very complicated self-consistency condition on $F(x)$ which has not yet been solved.

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APPENDIX A

The lemma quoted in the text is a special case of the following lemma. Given

$$\phi_n(z) = \chi(z)$$

Define

$$\tilde{\chi}(z) = \psi(\chi(\psi_{-1}(z))) \qquad \text{and} \qquad \tilde{\phi}(z) = \psi(\phi(\psi_{-1}(z)))$$

† In estimating effects of absorption by galactic dust we have used the semi-empirical relation

$$\bar{\rho} = \frac{\frac{4}{3}\Delta m a \rho}{1{\cdot}08 Q t}$$

where Δm is the extinction produced by a dust cloud of thickness t, $a \sim 3 \times 10^{-5}$ cm $\rho \sim 1$ gm/c.c., $Q \sim 1$, $\bar{\rho}$ is the mean density of the cloud. (See Hodge, 1966.)

where $\psi(z)$ is an arbitrary invertible function. Then it follows that

$$\tilde{\phi}_n(z) = \tilde{\chi}(z)$$

The proof of this lemma is very straightforward for n integer. It is not very difficult either for n rational. For arbitrary n the lemma then follows from suitable continuity requirements.

Taking $n=2$, $\psi(z) = z - a$ so that $\psi_{-1}(z) = z + a$ we get the special case of the lemma used above.

APPENDIX B

Derivation of

$$\frac{\partial^2 \phi_n(x)}{\partial n^2}\frac{\partial \phi_n(x)}{\partial x} - \frac{\partial^2 \phi_n(x)}{\partial n\,\partial x}\frac{\partial \phi_n(x)}{\partial n} = 0$$

In the following discussion we shall write $\phi_n(x)$ as $\phi(x,n)$ for the sake of notational convenience. Differentiation with regard to the first and second arguments of $\phi(x,n)$ will be denoted respectively as $\partial\phi/\partial u$, $\partial\phi/\partial v$.

Now

$$\phi(\phi(x,n), m-n) = \phi(x,m) \tag{B.1}$$

Differentiating with regard to n, we have

$$\frac{\partial\phi(\phi(x,n), m-n)}{\partial u}\frac{\partial\phi(x,n)}{\partial n} - \frac{\partial\phi(\phi(x,n), m-n)}{\partial v} = 0 \tag{B.2}$$

Differentiating (B.2) with regard to m,

$$\frac{\partial^2\phi(\phi(x,n), m-n)}{\partial v\,\partial u}\frac{\partial\phi(x,n)}{\partial n} - \frac{\partial^2\phi(\phi(x,n), m-n)}{\partial v^2} = 0 \tag{B.3}$$

Differentiating (B.2) with regard to x,

$$\frac{\partial^2\phi(\phi(x,n), m-n)}{\partial u^2}\frac{\partial\phi(x,n)}{\partial n}\frac{\partial\phi(x,n)}{\partial x} + \frac{\partial\phi(\phi(x,n), m-n)}{\partial u}$$

$$\frac{\partial^2\phi(x,n)}{\partial x\,\partial n} - \frac{\partial^2\phi(\phi(x,n), m-n)}{\partial u\,\partial v}\frac{\partial\phi(x,n)}{\partial x} = 0 \tag{B.4}$$

Differentiating (B.2) with regard to n,

$$\frac{\partial^2\phi(\phi(x,n), m-n)}{\partial u^2}\left(\frac{\partial\phi(x,n)}{\partial n}\right)^2 - \frac{\partial^2\phi(\phi(x,n), m-n)}{\partial v\,\partial u}\frac{\partial\phi(x,n)}{\partial n}$$

$$+\frac{\partial\phi(\phi(x,n), m-n)}{\partial u}\frac{\partial^2\phi(x,n)}{\partial n^2} - \frac{\partial^2\phi(\phi(x,n), m-n)}{\partial u\,\partial v}\frac{\partial\phi(x,n)}{\partial n}$$

$$+\frac{\partial^2\phi(\phi(x,n), m-n)}{\partial v^2} = 0 \tag{B.5}$$

Using equation (B.3) the last two terms on the left-hand side of equation (B.5) cancel and we get

$$\frac{\partial^2 \phi(\phi(x,n), m-n)}{\partial u^2}\left(\frac{\partial\phi(x,n)}{\partial n}\right)^2 - \frac{\partial^2 \phi(\phi(x,n), m-n)}{\partial v\, \partial u}\frac{\partial\phi(x,n)}{\partial n}$$
$$+\frac{\partial\phi(\phi(x,n), m-n)}{\partial u}\frac{\partial^2 \phi(x,n)}{\partial n^2} = 0 \qquad \text{(B.6)}$$

Multiplying equation (B.4) by $(\partial\phi(x,n))/\partial n$, equation (B.6) by $(\partial\phi(x,n))/\partial x$ and subtracting we get

$$\frac{\partial\phi(\phi(x,n), m-n)}{\partial u}\left(\frac{\partial^2 \phi(x,n)}{\partial x\,\partial n}\frac{\partial\phi(x,n)}{\partial n} - \frac{\partial^2 \phi(x,n)}{\partial n^2}\frac{\partial\phi(x,n)}{\partial x}\right) = 0$$

Dropping the factor $(\partial\phi(\phi(x,n), m-n))/\partial u$ which cannot vanish (for all m) for a non-trivial ϕ, we finally get

$$\frac{\partial^2 \phi(x,n)}{\partial x\,\partial n}\frac{\partial\phi(x,n)}{\partial n} - \frac{\partial^2 \phi(x,n)}{\partial n^2}\frac{\partial\phi(x,n)}{\partial x} = 0$$

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